Chapter 2.

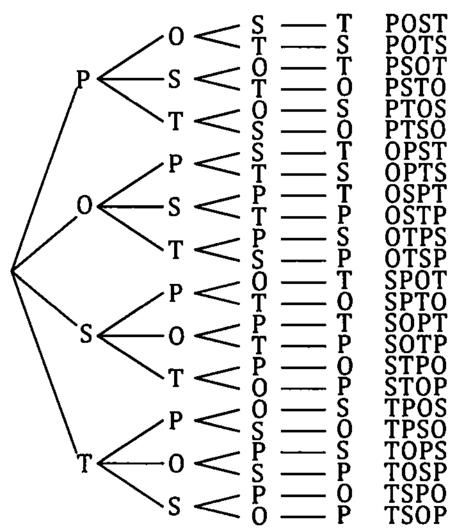
Counting.

Permutations (arrangements).

Asked to determine the number of possible **arrangements** (or **permutations**) there are of the letters of the word POST, with each letter being used once, we could list them:

POST	POTS	PSOT	PSTO	PTOS	PTSO
OPST	OPTS	OSPT	OSTP	OTPS	OTSP
SPOT	SPTO	SOPT	SOTP	STPO	STOP
TPOS	TPSO	TOPS	TOSP	TSPO	TSOP

to arrive at the answer of 24.



Alternatively we could choose a tree diagram form of display, as shown on the right, to again arrive at the answer of 24

However this listing is tedious, even with just the four letters of the word POST, so instead we develop techniques for counting the number of possible arrangements without having to list them all. Hence the title of this chapter, *counting*.

One useful principle we can use in this situation is the **multiplication principle**:

> If there are a ways an activity can be performed, and for each of these there are b ways that a second activity can be performed after the first, and for each of these there are c ways that a third activity can be performed after the second, and so on, then there are $a \times b \times c \times ...$ ways of performing the successive activities.

In the situation given above there were 4 choices of first letter, followed by 3 choices of second letter etc.

N$^{o.}$ of ways for each letter			
1^{st}	2^{nd}	3^{rd}	4^{th}
4	3	2	1

$$\text{Number of arrangements} = 4 \times 3 \times 2 \times 1 = 24$$

Factorial notation.

Use of the multiplication principle frequently involves us in evaluating expressions like

$$2 \times 1$$
$$3 \times 2 \times 1$$
$$4 \times 3 \times 2 \times 1$$
$$5 \times 4 \times 3 \times 2 \times 1$$
$$6 \times 5 \times 4 \times 3 \times 2 \times 1$$

We write n!, pronounced "n **factorial**", to represent

$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ where n is a positive integer.

For example

$$\begin{aligned} 3! &= 3 \times 2 \times 1 \\ &= 6 \end{aligned}$$

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

$$\begin{aligned} 10! &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 3\,628\,800 \end{aligned}$$

3!	6
5!	120
10!	3628800

Example 1

Evaluate (a) 6! (b) $5! \div 3!$ (c) $100! \div 98!$

(a)
$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

(b)
$$\begin{aligned} 5! \div 3! &= \frac{5 \times 4 \times 3!}{3!} \\ &= 5 \times 4 \\ &= 20 \end{aligned}$$

(c)
$$\begin{aligned} 100! \div 98! &= \frac{100 \times 99 \times 98!}{98!} \\ &= 100 \times 99 \\ &= 9\,900 \end{aligned}$$

Using this factorial notation we can say:

There are n! ways of arranging n different objects in a row. I.e. n! permutations.

Example 2

How many different five letter "words" can be formed using the letters of the word MATHS if (a) each letter is used just once,
(b) each letter can be used any number of times.

(a) The first letter can be chosen in 5 ways, the second can then be chosen in 4 ways, the third in 3 ways etc.

Total number of words $= 5 \times 4 \times 3 \times 2 \times 1$
$= 120$

Nº. of ways for each letter				
5	4	3	2	1

(b) The first letter can be chosen in 5 ways, the second can then be chosen in 5 ways, the third in 5 ways etc.

Total number of words $= 5 \times 5 \times 5 \times 5 \times 5$
$= 3125$

5	5	5	5	5

Permutations of objects from a group of objects, all different.

In our consideration of the arrangements of the letters of the word POST, on an earlier page, we were using *4 letters* to make *4 letter* words.

Similarly in example 2 above we were arranging *5 letters* to make *5 letter* words.

Both of these situations involved using all of the available letters in each word.

How then do we determine the number of 5 letter words that can be created if we have more than 5 available? Again the multiplication principle comes to our aid.

Let us consider the case of making 2 letter words when the 5 letters of the word CAKES are available, with no repeat use of letters allowed.

We have **5** choices for first letter: C, A, K, E, S.

Having chosen the first letter we then have 4 choices of second letter.

Total number of choices $= 5 \times 4$
$= 20$

The complete listing is shown on the right

CA	AC	KC	EC	SC
CK	AK	KA	EA	SA
CE	AE	KE	EK	SK
CS	AS	KS	ES	SE

Thus given n different objects, and wanting to arrange r of them:

The 1st could be chosen in n ways, the 2nd in (n − 1) ways, the 3rd in (n − 2) ways ..... until we get to the rth object which could be chosen in (n − r + 1) ways.

$$\text{Number of permutations} = n \times (n-1) \times (n-2) \times \ldots\ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Hence:

The number of permutations of r objects taken from n different objects is

$$\frac{n!}{(n-r)!}.$$

We write this as ^nP_r.

Example 3

How many different three letter "words" can be formed using the letters of the word MAKER if each letter is used just once.

The first letter can be chosen in 5 ways, the second can then be chosen in 4 ways and the third in 3 ways.

N^{o.} of ways for each letter		
5	4	3

$$\text{Total number of words} = 5 \times 4 \times 3 = 60$$

Or, using the ^nP_r notation:

Number of 3 letter arrangements chosen from 5 different letters is $^5P_3$.

$$^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

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5P3
                    60
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Example 4

One student is to be selected from the six students

Armand, Chris, Jennifer, Kate, Tony, Varun,

to be student of the week for week 1. A different student from the group is then selected to be student of the week for week 2 and another is selected for week 3. In how many ways can this be done?

The student for week 1 can be chosen from any one of the six. The student for week two can then be chosen from the remaining 5 and the student for week 3 can be chosen from the remaining 4. Total number of ways $= 6 \times 5 \times 4 = 120$

N^{o.} of ways for each week		
6	5	4

Note: ^nP_r gives the number of permutations of r objects taken from n different objects. However we know that there are n! permutations if we use all n objects.

Thus $\quad ^nP_n = n!$

I.e. $\quad \dfrac{n!}{(n-n)!} = n!$

Thus we need to define (n − n)!, i.e. 0!, to equal 1. Check that your calculator agrees with this value for zero factorial.

Exercise 2A

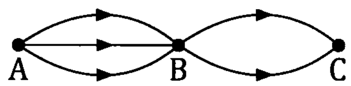
1. Evaluate each of the following, without the assistance of a calculator.

   (a) $3!$ (b) $3! + 2!$ (c) $(3 + 2)!$

   (d) $\frac{11!}{10!}$ (e) $\frac{11!}{9!}$ (f) $\frac{6!}{4!\,2!}$

   (g) ${}^5P_2$ (h) ${}^7P_3$ (i) ${}^8P_2$

2. Three roads are available for the journey from town A to town B and two are available from town B to town C.



   How many different routes are possible from A to C, via B?

3. How many different ham rolls are possible if we have four different types of roll, just one type of ham, lettuce or no lettuce, mustard or no mustard?

4. A father places six marbles, each of a different colour, in a row on the floor. His two year old daughter tells him he has put them in the wrong order! How many different orders are possible, other than the one he has already used?

5. Each day a student can make the journey to school in one of three ways:

   By bus, by bike or on foot.

   How many different ways can the student arrange his travel to school in the course of one week (five school days)?
   For example one arrangement could be:

   bus, bike, foot, foot, bike.

6. How many different seven letter "words" can be formed using the letters of the word FASHION if (a) each letter is used just once in each word,
   (b) each letter can be used any number of times.

7. How many different five letter "words" can be formed using the letters of the word FASHION if (a) each letter is used just once in each word,
   (b) each letter can be used any number of times.

8. The names of the fifteen finalists in a competition are each written on a piece of paper and the fifteen pieces are placed in a hat. The first name drawn out then wins the first prize, the second out wins second prize and the third out wins third prize. How many different ways can the three prizes be awarded if, after a name is drawn from the hat, it is (a) returned to the hat,
   (b) not returned to the hat?

9. A science teacher develops a computer testmaker program for her class. The program contains a bank of science questions from which a number of different tests can be constructed. The questions are classified in terms of difficulty using a ten point scale, I to X. To run the program the user specifies x, the number of questions in the test ($x \leq 10$) and the program creates a test with question 1 of difficulty level I, question 2 of difficulty level II, question 3 of difficulty level III etc up to question x of difficulty level x. The number of questions available at each level of difficulty is:

Level	I	II	III	IV	V	VI	VII	VIII	IX	X
No. of qns.	12	10	10	6	8	4	9	6	5	5

How many different tests can be created if the test is to contain
(a) five questions, (b) eight questions, (c) ten questions.

10. A "straight exacta" bet on a horserace requires the person making the bet to state, in order, the horse they think will come first and the horse they think will come second. How many different "straight exactas" could be made on a race involving 12 horses?

11. How many different four digit security numbers can be made using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if the numbers may start with 0 but must not contain any repeated digits. i.e. numbers like 4436 (two 4s), 1281 (two 1s), 3533 (three 3s) are not permitted.
(Security numbers are frequently referred to as PINs. Why?)

12. Let us suppose that a "first six" bet on a horserace requires the person making the bet to state, in the correct order, the horses that will finish in the first six places. How many different "first six" bets could be made on a race involving 12 horses?

13. A test consists of ten multiple choice questions with each question having four possible choices. Assuming all ten questions are attempted, in how many ways can the multiple choice test be answered?

14. A ballot paper lists fifteen candidates and requires the voter to write 1st next to one of these fifteen, 2nd against another and third against another. In how many ways can this be done?

15. A television soccer program invites viewers to enter the *Goal of the month* competition by listing a 1st, 2nd and 3rd place from a choice of eight goals. How many different entries are possible?
How many different entries are possible if instead all eight had to be listed in order of preference?

16. A test comprises ten questions each requiring a response of either yes or no. Assuming all questions are responded to, in how many ways can the test be answered?

Permutations of objects, not all different.

Consider now arranging the following into a row:

$a_1, \quad a_2, \quad a_3, \quad b.$

This involves arranging 4 different objects so there will be 4!, or 24, arrangements:

$a_1\,a_2\,a_3\,b$	$a_1\,a_2\,b\,a_3$	$a_1\,b\,a_2\,a_3$	$b\,a_1\,a_2\,a_3$
$a_1\,a_3\,a_2\,b$	$a_1\,a_3\,b\,a_2$	$a_1\,b\,a_3\,a_2$	$b\,a_1\,a_3\,a_2$
$a_2\,a_1\,a_3\,b$	$a_2\,a_1\,b\,a_3$	$a_2\,b\,a_1\,a_3$	$b\,a_2\,a_1\,a_3$
$a_2\,a_3\,a_1\,b$	$a_2\,a_3\,b\,a_1$	$a_2\,b\,a_3\,a_1$	$b\,a_2\,a_3\,a_1$
$a_3\,a_1\,a_2\,b$	$a_3\,a_1\,b\,a_2$	$a_3\,b\,a_1\,a_2$	$b\,a_3\,a_1\,a_2$
$a_3\,a_2\,a_1\,b$	$a_3\,a_2\,b\,a_1$	$a_3\,b\,a_2\,a_1$	$b\,a_3\,a_2\,a_1$

However, if the three letter "a"s were indistinguishable the above list would shrink to show just 4 arrangements

a a a b	a a b a	a b a a	b a a a

Each of the 3! (= 6) arrangements of a_1, a_2 and a_3 forming a column in the first listing now giving rise to just one arrangement in the second list.

Hence there are $\frac{4!}{3!}$ arrangements of the letters a, a, a, b.

If n objects contain p of one kind, q of another, r of another etc, then there are

$$\frac{n!}{p!\ q!\ r! \ldots}$$

arrangements of the n objects.

Example 5

How many ways can the nine letters of the word ISOSCELES be arranged in a row?
How many of these start with the L?

The word ISOSCELES involves 9 letters including 3 Ss and 2 Es.

$$\text{Number of arrangements} = \frac{9!}{3!\,2!} = 30240$$

There are 30240 arrangements of the nine letters of the word ISOSCELES.

If we start with the L then we are arranging the other 8 letters, which include 3 Ss and 2 Es, and then placing an L at the front of each arrangement.

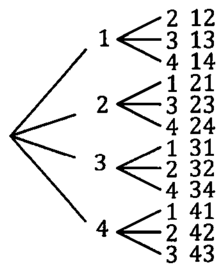
$$\text{Number of arrangements starting with the L} = \frac{8!}{3!\,2!} = 3360$$

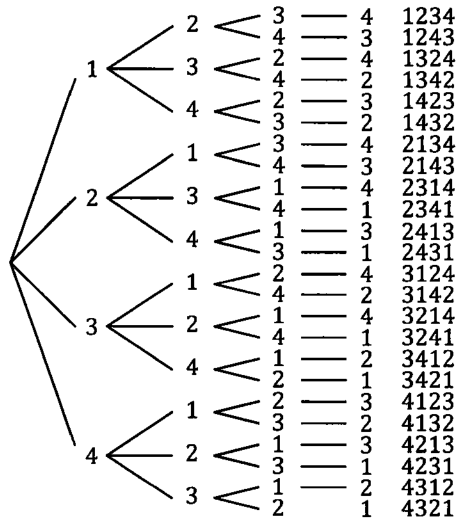
There are 3360 arrangements that start with the L.

Addition principle.

Consider the following question:
How many code numbers can be made from the digits 1, 2, 3, and 4 if the code numbers can either be two digit numbers or four digit numbers but no code may use the same digit more than once?

The 36 possible code numbers are shown in the two tree diagrams on the right.





However we know that to work out the number of possible 2 digit codes and the number of possible 4 digit codes we do not need to display them all.

Instead we can use *multiplicative reasoning*:

Number of 2 digit codes $= 4 \times 3$
$= 12$

Number of 4 digit codes $= 4 \times 3 \times 2 \times 1$
$= 24$

Hence the total number of possible 2 or 4 digit codes will be 12 + 24, i.e. 36.

Notice that in the last step we added the number of two digit codes to the number of four digit codes. We used *additive reasoning.*

This reasoning is formalised in the *addition principle*:

The addition principle:

> If there are a ways event A can occur and b ways that event B can occur then, provided A and B are mutually exclusive (i.e. A and B cannot occur together), $a + b$ is the number of ways either A or B can occur.

Example 6

A three character code can either consist of three digits from 1, 2, 3, 4, and 5 or it can consist of three letters from A, B, C, D and E. How many codes are possible if digits can be used more than once in a code but letters cannot?

Number of possible three digit codes $= 5 \times 5 \times 5$
$= 125$

Nº of ways for each digit		
5	5	5

Number of possible three letter codes $= 5 \times 4 \times 3$
$= 60$

Nº of ways for each letter		
5	4	3

These events are mutually exclusive because a three character code cannot be both three digit and three letters. Thus 185 (= 125 + 60) codes can be formed.

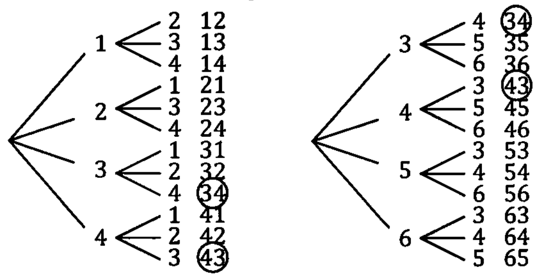
Care must be taken regarding the requirement that the two events cannot occur together, i.e. the *mutually exclusive* requirement. In the previous situation that prompted the tree diagrams, and in example 6, the two types of code were mutually exclusive. A code could not be both 2 digits and 3 digits, and in example 6 a 3 digit code code could not be both three digits and 3 letters. The two events were mutually exclusive. If this requirement for *mutually exclusivity* is not met we will find we are counting some arrangements twice.

For example suppose instead the two events involved making a two digit code number from the digits 1, 2, 3, 4, and making a two digit code number from the digits 3, 4, 5, 6, no repeat digits allowed. These events are not mutually exclusive because the code numbers 34 and 43 appear in both, as we can see from the tree diagrams below.

In this case Number of codes $\neq$ 12 + 12

Instead: Number of codes = 22

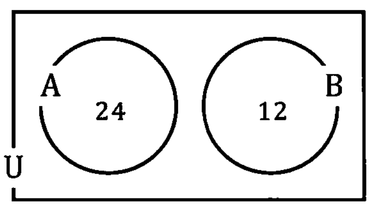
i.e. 12 + 12 − 2 (the 2 being subtracted to compensate for the two repeated codes)



Inclusion – exclusion principle.

The 2 digit code and 3 digit code situation shown as tree diagrams on the previous page involved mutual exclusivity thus allowing us to add together the numbers 12 and 24. The Venn diagram on the right shows this mutually exclusive situation.

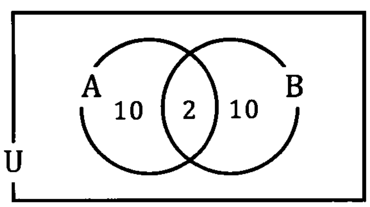
$$n(A \cup B) = n(A) + n(B) = 24 + 12 = 36$$



The Venn diagram on the right shows the non mutually exclusive situation shown in the tree diagrams considered at the top of this page. Now our addition uses the more general rule for the union of two sets:

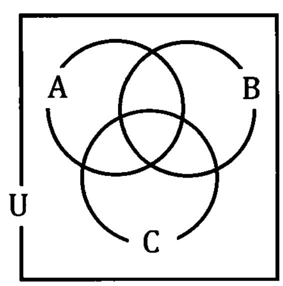
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\text{Number of codes} = 12 + 12 - 2 = 22$$



Extending this idea to three sets gives the following expression for $n(A \cup B \cup C)$:

$$n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$



Notice that in this rule we first

include the numbers in each set,

and then take away (**exclude**) the numbers in the "two set intersections",

and then **include** the number in the "three set intersection".

This alternating "inclusion – exclusion" idea can be extended to 4 or more sets and is known as the **inclusion – exclusion principle**.

Example 7

(a) How many multiples of 2 are there between 1 and 89 ?
(b) How many multiples of 3 are there between 1 and 89 ?
(c) How many numbers between 1 and 89 are multiples of 2 or 3 ?

Note: In mathematics we take the word "or" to mean "one or the other or both". In other words we take "or" to mean "at least one of".
Thus for two events A and B we take "A or B" to mean $A \cup B$.

(a) The multiples of 2 between 1 and 89 are: 2, 4, 6, 8, ... 88.
44 numbers in total.

(b) The multiples of 3 between 1 and 89 are: 3, 6, 9, 12, ... 87.
29 numbers altogether.

(c) The numbers between 1 and 89 that are multiples of both 2 and 3 are:
6, 12, 18, 24, ... 84
14 numbers altogether.
Thus the required number $= 44 + 29 - 14$
$= 59$
There are 59 numbers between 1 and 89 that are multiples of 2 or 3.

Exercise 2B

The first six questions involve: Arrangements of objects, not all different.

1. How many ways can the 6 letters of the word REPEAT be arranged in a row?

2. How many ways can the 7 letters of the word CLASSES be arranged in a row?

3. How many ways can the 7 letters of the word TROTTER be arranged in a row?

4. How many 12 letter permutations are there of the word PERMUTATIONS ?

5. How many ways can the 11 letters of the word MISSISSIPPI be arranged in a row. How many of these start with the M?

6. How many ways can the 10 letters of the word WOLLONGONG be arranged in a row?
How many of these start with the W?
How many do not start with the W?

The remaining questions involve an understanding of : n(A ∪ B).

7. How many two or three digit numbers can be formed using the digits 1, 2, 3, 4, 5 if no digit may be used more than once in a number?

8. How many two or three letter codes can be formed using letters from the alphabet and allowing any letter to be used more than once?

9. How many two or three letter codes can be formed using letters from the alphabet if each code must not feature any letter more than once?

10. The digits 1, 2 and 3 are to be used to make two digit numbers and three digit numbers. How many such numbers are possible if
    (a) repeated use of digits in a number is permitted,
    (b) repeated use of digits in a number is not permitted?

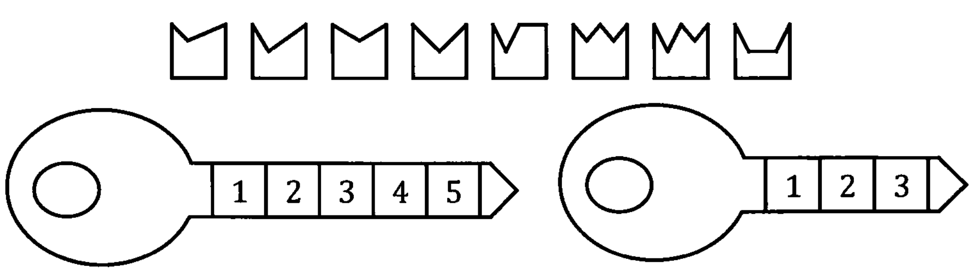
11. The digits 1, 2, 3, 4 and 5 are to be used to make two digit numbers and three digit numbers. How many such numbers are possible if
    (a) repeated use of digits in a number is permitted,
    (b) repeated use of digits in a number is not permitted?

12. A four character code can either consist of four digits from 1, 2, 3, 4, 5, or it can consist of four letters from A, B, C, D, E, F and G. How many codes are possible if digits can be used more than once in a code but letters cannot?

13. A three letter code comprising different letters is to be made either by arranging three letters chosen from the word MATHS or by obtaining the first letter in the arrangement from this source and then obtaining the next two letters from the word FUN. How many such codes are possible?

14. Sanshi is at the race track and wants to place one more bet, either on race 8 or race 9, but not on both. He is not sure whether to randomly predict 1st, 2nd and 3rd for race 8, in which there are 8 horses racing or randomly predict a 1st and 2nd for race 9 with 12 horses racing.
    How many different bets does this choice involve altogether?

15. A lock making company makes a number of different locks each with their own key design. The keys are made by cutting one of eight "cut styles" in each of the five positions in the long key base model or in each of the three positions in the short key base model.



How many different keys are possible in each of the following situations:
(a) Each key can feature the same "cut style" in more than one of its positions.
(b) Each key must not feature any one "cut style" more than once.

16. Repeat question 15 but now include the possibility of each position in a key remaining uncut.

17. A two digit code number is to be made either by using two different digits from the digits 1, 2, 3, 4, 5, 6 or by using two different digits from the digits 6, 7, 8.
How many different two digit codes are possible?

18. A national credit company decides to give each of its employees a security card for computer access. When the card is placed into the reader the employee will be asked to type in their code "word". These code words will consist of four different letters either all taken from the word CREDIT or all taken from the word COMPANY. How many different code words are possible?

19. When choosing her new car Shahani has narrowed her choice down to two possible models: *The Nifty Townabout* or *The Sedate Tourer*.
The *Nifty* is available in 4 colours, 2 engine sizes, with or without air conditioning, with or without automatic transmission and with or without power steering.
The *Sedate* is available in 5 colours, 3 engine sizes and all models feature air conditioning, automatic transmission and power steering.
How many "different" cars is Shahani actually considering?

20. Code numbers consisting of two different digits are to be made.
The numbers will consist of two digits chosen from the digits 1, 2, 3, 4, 5 or two digits chosen from the digits 4, 5, 6, 7.
How many different two digit codes are possible?

21. Code numbers consisting of three different digits are to be made.
The numbers will consist of three digits chosen from the digits 1, 2, 3, 4, 5 or three digits chosen from the digits 3, 4, 5, 6.
How many different three digit codes are possible?

22. (a) How many multiples of five are there between 1 and 999?
(b) How many multiples of seven are there between 1 and 999?
(c) How many numbers between 1 and 999 are multiples of five or seven? (Remember that we take "or" to mean "at least one of".)

23. Use the inclusion – exclusion principle for three sets, i.e.
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
to determine $|A \cup B \cup C|$ given the following information:

$n(A) = 27$ $n(B) = 17$ $n(C) = 29$
$n(A \cap B) = 12$ $n(A \cap C) = 18$ $n(B \cap C) = 10$
$n(A \cap B \cap C) = 7$

Check your answer by drawing a Venn diagram of the situation.

24. How many numbers between 1 and 101 are multiples of 2, 3 or 5.

25. How many numbers between 1 and 1001 are multiples of 3, 10 or 25.

26. Use the alternating inclusion, exclusion nature of the inclusion - exclusion principle to write the rule for | A ∪ B ∪ C ∪ D | for the four sets A, B, C and D.

Arrangements of objects with some restriction imposed.
Some of the questions encountered so far have involved arrangements with some restriction imposed, for example arranging the letters of the word ISOSCELES with the restriction that the first letter must be the L. This section considers such ideas further.

I. Multiplicative reasoning.
If asked to determine the number of 3 letter codes that can be formed using the letters A, B, C, D, E, given that a code cannot use a letter more than once, we would proceed as follows:

There are five choices for the first letter, four for the second and three for the third.

Nº of ways for each letter		
5	4	3

Number of three letter codes $= 5 \times 4 \times 3$

$= 60$ i.e. $^5P_3$

Suppose now that some restriction is involved, for example that the middle letter must be a vowel. How many different three letter arrangements are there now?

In such cases we can still use multiplicative reasoning to determine the total number of arrangements but it is advisable to **consider the restriction first**, as follows.

There are two choices for the middle letter, A or E. There then remain four choices for the first, B, C, D and whichever of A and E was not used for the middle letter, and then three for the last.

Total number of arrangements $= 4 \times 2 \times 3$ i.e. 24.

Nº of ways for each letter		
?	2	?

4	2	3

Example 8
How many different <u>five</u> letter "words" can be formed using the letters of the word FASHION if the middle letter in each arrangement must be the S and no word may feature a letter used more than once.

There is only one choice for the middle letter, an S.

There then remain six choices for the first, F, A, H, I, O, N, five for the second, four for the fourth and three for the last.

Total number of arrangements $= 6 \times 5 \times 1 \times 4 \times 3$

$= 360$

Nº of ways for each letter				
?	?	1	?	?

6	5	1	4	3

Example 9

(a) How many five digit <u>even</u> numbers can be made using the digits 3, 4, 5, 6 and 7 if no digit may feature more than once in a number.

(b) How many of the numbers from (a) are greater than 70 000?

(a) Start by choosing the last digit, which must be either the 4 or the 6, to ensure an even number:

Nº. of ways for each digit				
?	?	?	?	2

Left hand digit is then chosen from remaining 4, next digit from remaining 3 and so on:

4	3	2	1	2

The number of possible five digit even numbers $= 4 \times 3 \times 2 \times 1 \times 2$
$= 48$

(b) The last digit must be either the 4 or the 6 and the first digit must be the 7.

Nº. of ways for each digit				
1	?	?	?	2

The next digit is then chosen from the remaining 3 and so on:

1	3	2	1	2

Number that are even and greater than 70 000 $= 1 \times 3 \times 2 \times 1 \times 2$ i.e. 12.

Example 10

A security code consists of four digits chosen from 0, 1, 2, ... , 9 followed by a capital letter chosen from A, B, C, ... , Z. For example

3	7	0	9	E

How many such codes are possible in each of the following cases?

(a) No digit is to feature more than once in a code.

(b) There is no restriction on the number of times a digit may feature in a code.

(c) The code must not start with zero and no digit must be used more than once in a code.

(d) The code must not start with zero, must end with a vowel, and no digit may feature more than once in a code.

(a) If we start by choosing the first digit and then work across, the number of ways each entry can be chosen will be:

10	9	8	7	26

The number of possible codes is $10 \times 9 \times 8 \times 7 \times 26 = 131\,040$

(b) Number of ways each entry can be chosen will be:

10	10	10	10	26

The number of possible codes is $10 \times 10 \times 10 \times 10 \times 26 = 260\,000$

(c) The first digit can be chosen in 9 ways (zero is <u>not</u> allowed), the second can then be chosen in 9 ways (now zero <u>is</u> allowed), the third in 8 etc. The number of ways each entry can be chosen will be:

9	9	8	7	26

The number of possible codes is $9 \times 9 \times 8 \times 7 \times 26 = 117\,936$

(d) Number of ways each entry can be chosen will be:

9	9	8	7	5

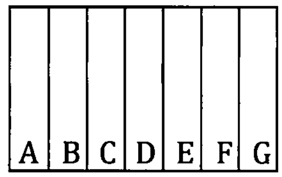
The number of possible codes is $9 \times 9 \times 8 \times 7 \times 5 = 22\,680$

Example 11

Seven files, A, B, C, D, E, F and G are to be arranged on a shelf.

(a) In how many ways can this be done?

(b) In how many of these arrangements is file A next to file B?



(a) The first file can be chosen in 7 ways, the next in 6, the next in 5 and so on.
Total number of arrangements $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ i.e. 7!, or 5040.

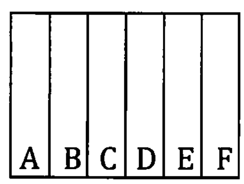
(b) If we imagine files A and B tied together we then have six things to arrange. However, A and B could be tied together in 2 ways (AB or BA).
Total number of arrangements $= (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 2$ i.e. $6! \times 2$, or 1 440.

Exercise 2C

1. How many different <u>five</u> letter "words" can be formed using the letters of the word GREAT if the second letter in each arrangement must be the G and
   (a) no word may feature a letter used more than once,
   (b) letters may be used more than once in a word.

2. (a) How many six digit <u>odd</u> numbers can be made using the digits 1, 2, 3, 4, 5 and 6 if no digit may feature more than once in a number.
   (b) How many of the numbers from (a) are greater than 600 000?

3. Six files, A, B, C, D, E and F are to be arranged on a shelf.
   (a) In how many ways can this be done?
   (b) In how many of the arrangements is file D next to file F?
   (c) In how many of the arrangements are files ABC together in that order?
   (d) In how many of the arrangements are files ABC together but in no particular order?



4. How many ways can the five letters

X, A, E, Q, R

be written in a line if no letter can be used more than once?
How many of these arrangements of the five letters
(a) start with a consonant, (b) start with a vowel?

5. In how many different orders can the five names

Alex, Dennis, Jack, Jill, and Kris be written?

In how many of these is (a) Jack listed first,
(b) Jill listed second,
(c) Jack listed first and Jill listed second?

6. (a) How many seven digit numbers can be made using the digits 1, 2, 3, 4, 5, 6 and 7 if no digit may be used more than once in a number.
(b) How many seven digit *even* numbers can be made using the digits 1, 2, 3, 4, 5, 6 and 7 if no digit may be used more than once in a number.
(c) How many of the even numbers from (b) are bigger than 7 000 000?

7. A restaurant offers a special 3 course lunch deal. Customers choose one starter, one main course and one dessert from:

Starter	Main Course	Dessert
Prawn Cocktail	Chicken Kiev	Apple pie
Paté	Steak	Ice cream
Soup	Lasagne	
	Cottage Pie	

(a) How many different three course meals are possible?
(b) How many of these involve Lasagne?
(c) How many involve Lasagne and Ice cream?

8. A security code consists of three digits chosen from 0, 1, 2, ... , 9 followed by two letters chosen from A, B, C, ... , Z. For example

2	4	1	Z	E

How many such codes are possible in each of the following cases?
(a) No digit is to feature more than once in a code but the letters can be repeated.
(b) There is no restriction on the number of times each digit may feature in a code but no repeat letters are allowed.
(c) The code must not start with zero and no digit nor letter may feature in a code more than once.
(d) The code must not start with zero, letters and digits can be used more than once but each code must end with a vowel.

9. A coin is tossed, then a normal die is rolled, then another coin is tossed and another normal die is rolled.

   One possible sequence of results is H, 5, T, 1.

   (a) How many possible sequences are there?

   In how many of the sequences do

   (b) the two dice show the same number?

   (c) the two coins show the same result?

10. A security code consists of three letters chosen from the 26 in the alphabet followed by two digits chosen from 0, 1, 2, ... , 9.

    For example

Z	E	K	4	3

    How many such codes are possible in each of the following cases?

    (a) There are no restrictions on the choice of letters and digits.

    (b) No letter and no digit may feature more than once in the code.

    (c) The initial letter must not be a vowel and no letters or digits are to be used more than once.

    (d) The first and third letters must be the same as each other and different to the second, and the two digits must be the same as each other.

    (e) There is no restriction on the two digits but the letters must be consecutive letters of the alphabet and must feature in alphabetical order ("reverse alphabetical" not permitted).

    For example

P	Q	R	5	2

    is acceptable.

    (f) The final digit must be one more than the digit before it and the letters must be consecutive letters of the alphabet featuring in alphabetical order ("reverse alphabetical" not permitted).

    For example

L	M	N	5	6

    is acceptable.

11. Ten books are to be arranged on a shelf. Three of the ten books are by one author and the other books are all by different authors.

A1	A2	A3	B	C	D	E	F	G	H

    How many arrangements are possible if the three by the same author

    (a) need not be kept together,

    (b) must be kept together and in a particular order,

    (c) must be kept together but in no particular order,

    (d) must be kept together at the left end of the shelf and in a particular order?

Arrangements of objects with some restriction imposed.
II. Additive and multiplicative reasoning.

As has been mentioned before, the use of *additive reasoning* needs care. If we are adding the number of ways an event A can occur to the number of ways an event B can occur this will only give the number of ways A or B can occur if A and B cannot occur together, i.e. provided A and B are **mutually exclusive**. If this is not the case the addition will mean that some events will be counted more than once. In such cases we can either break the task into separate mutually exclusive situations, and then it will be safe to add, or we can adjust for the "double counting" by using $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

(Remember in mathematics we interpret "A or B" to mean "at least one of".)

For example, suppose we are forming four letter codes from the letters

A, B, C, D, E, F,

with no code using a letter more than once.

No. of ways for each letter			
1	5	4	3

1	5	4	3

Number of codes starting with an A $= 1 \times 5 \times 4 \times 3$
$= 60$

Number of codes starting with an F $= 1 \times 5 \times 4 \times 3$
$= 60$

Number starting with an A or an F $= 60 + 60$
$= 120$

The use of additive reasoning is valid because a code cannot both start with an A and start with an F. The two events are mutually exclusive.

Contrast this with the "start with an A or end with an F" situation.

No. of ways for each letter			
1	5	4	3

5	4	3	1

Number of codes starting with an A $= 1 \times 5 \times 4 \times 3$
$= 60$

Number of codes ending with an F $= 5 \times 4 \times 3 \times 1$
$= 60$

But the number of codes starting with A or ending with F $\neq 60 + 60$ because this would count codes like ABCF, ACDF, ADCF, etc., twice. The events are not mutually exclusive as a code can start with an A and at the same time end with an F.

Instead we either use $n(A \cup B) = n(A) + n(B) - n(A \cap B)$:

Number of codes starting with an A and end with an F $= 1 \times 4 \times 3 \times 1$
$= 12$

Thus number that start with an A or end with an F $= 60 + 60 - 12$
$= 108$

Or we consider the mutually exclusive situations:

Start with an A and end with an F $1 \times 4 \times 3 \times 1 = 12$
Start with an A and not end with an F: $1 \times 4 \times 3 \times 4 = 48$
Do not start with an A but end with an F: $4 \times 4 \times 3 \times 1 = 48$
Thus number that start with an A or end with an F $= 12 + 48 + 48$
$= 108$, as before.

Example 12

How many three digit codes can be made using the digits 0, 1, 2, 3, 4, 5 if there are no restrictions on using a digit more than once?

How many of these codes
(a) start with a 3,
(b) start with a 4,
(c) do not start with a 4,
(d) start with a 3 or a 4,
(e) start with a 3 and end with a 4,
(f) start with a 3 or end with a 4.

Number of ways each digit of code can be chosen:

6	6	6

The number of possible codes is $6 \times 6 \times 6 = 216$

(a) If the code must start with a 3:

1	6	6

The number of possible codes is $1 \times 6 \times 6 = 36$

(b) If the code must start with a 4:

1	6	6

The number of possible codes is $1 \times 6 \times 6 = 36$

(c) If the code must not start with a 4:

5	6	6

The number of possible codes is $5 \times 6 \times 6 = 180$

Alternatively: 36 of the 216 codes start with a 4 therefore 180 (= 216 − 36) do not start with a 4. (This alternative method uses the idea of the *complementary* event.)

(d) If the code must start with a 3 or a 4:

2	6	6

The number of possible codes is $2 \times 6 \times 6 = 72$

Alternatively, using additive reasoning because start with 3 and start with a 4 are mutually exclusive:

Number of codes that start with a 3 or a 4 is $36 + 36 = 72$

(e) If the code must start with a 3 and end with a 4:

1	6	1

The number of possible codes is $1 \times 6 \times 1 = 6$

(f) If the code must start with a 3:

1	6	6

The number of possible codes is $1 \times 6 \times 6 = 36$

If the code must end with a 4:

6	6	1

The number of possible codes is $6 \times 6 \times 1 = 36$

"Start with 3" and "end with a 4" are not mutually exclusive. They can occur together, as in the code 314. The answer will not be 36 + 36.

Using $n(A \cup B) = n(A) + n(B) - n(A \cap B)$:

Number of codes starting with 3 or end with 4 is $36 + 36 - 6$
$= 66$

Or, considering "start with a 3 or end with a 4" as three mutually exclusive events:

- Start with a 3 and end with a 4:

1	6	1

- Start with a 3 and not end with a four:

1	6	5

- End with a 4 but not start with a 3:

5	6	1

Number of codes starting with 3 or end with 4 is $6 + 30 + 30$
$= 66$, as before.

Example 13

Five members of a basketball team all have to stand in a line for a photograph.
The players are: Alex, Keith, Mark, Rani and Steve.
How many arrangements are there in which

(a) Rani is in the middle,
(b) Alex is at the left end,
(c) Mark is at the right end,
(d) At least one of (b) and (c) occur,
(e) Keith and Steve are next to each other,
(f) Keith and Steve are not next to each other.

(a) Middle place filled in one way:

		1		

Then fill left most space from remaining 4 players, next from remaining 3 etc.:

4	3	1	2	1

Number of possible arrangements is $4 \times 3 \times 1 \times 2 \times 1 = 24$.

(b) Alex at left end then fill spaces from left:

1	4	3	2	1

Number of possible arrangements is $1 \times 4 \times 3 \times 2 \times 1 = 24$.

(c) Mark at right end then fill spaces from left:

4	3	2	1	1

Number of possible arrangements is $4 \times 3 \times 2 \times 1 \times 1 = 24$.

(d) Alex at left end and Mark not at right end:

1	3	2	1	3

Mark at right end and Alex not at left end:

3	3	2	1	1

Alex at left end and Mark at right end:

1	3	2	1	1

Number of possible arrangements is $18 + 18 + 6 = 42$.

Or: Add answers from (b) and (c) then subtract the "and situation" that would otherwise be counted twice: $24 + 24 - 6 = 42$, as before.

(e) "Tie" Keith and Steve together (2 ways). We now have 4 "things" to arrange.
Number of possible arrangements is $2 \times 4 \times 3 \times 2 \times 1 = 48$.

(f) There are $5 \times 4 \times 3 \times 2 \times 1$ (= 120) arrangements altogether and 48 have Keith and Steve together. Thus 72 (= 120 − 48) must have Keith and Steve not together.
There are 72 arrangements in which Keith and Steve are not together.

Note the use of the complementary event in part (f).

Example 14

(a) How many six digit even numbers can be made using the digits 1, 2, 3, 4, 5 and 6 if no digit may be used more than once in a number.

(b) How many of the numbers from (a) are bigger than 400 000?

(a) Start by choosing the last digit, which must be 2, 4 or 6, to ensure the number is even:

?	?	?	?	?	3

Left hand digit is then chosen from remaining 5, next digit from remaining 4 and so on:

5	4	3	2	1	3

The number of possible 6 digit even numbers $= 5 \times 4 \times 3 \times 2 \times 1 \times 3$
$= 360$

There are 360 six digit even numbers that can be made using the digits 1, 2, 3, 4, 5 and 6.

(b) The problem now is that if we choose the last digit from the 2, 4 or 6 the number of choices for the first digit is 3, if the last digit was the 2, and 2 if the last digit was the 4 or the 6.

To cope with this we consider mutually exclusive events.

Ending with 2 and bigger than 400 000:

3	4	3	2	1	1

Ending with 4 and bigger than 400 000:

2	4	3	2	1	1

Ending with 6 and bigger than 400 000:

2	4	3	2	1	1

The number of possible 6 digit even numbers > 400 000 $= 72 + 48 + 48$
$= 168$

There are 168 six digit even numbers bigger than 400 000 that can be made using the digits 1, 2, 3, 4, 5 and 6.

Exercise 2D

1. The digits 1, 2, 3, 4 and 5 are to be used to make three digit numbers and four digit numbers. How many such numbers are possible if
   (a) repeated use of digits in a number is permitted,
   (b) repeated use of digits in a number is not permitted,
   (c) repeated use of digits in a number is not permitted and only odd numbers are to be formed?

2. The letters U, V, W, X, Y and Z are to be used to make three letter and five letter code words. Determine how many such code words are possible in each of the following situations.
   (a) Repeated use of letters in a code is permitted,
   (b) Repeated use of letters in a code is not permitted,
   (c) Repeated use of letters is not permitted and no code word is allowed to start with the Z?

3. (a) How many seven digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6 and 7 if no digit may be used more than once in a number.
   (b) How many of the numbers from (a) are greater that 6 000 000?

4. In how many ways can the five digits 1, 2, 3, 4, 5 be arranged to form a five digit code number, with each code using all five digits once each?
   How many of these arrangements
   (a) start with 3,
   (b) end with 5,
   (c) start with 3 and end with 5,
   (d) start with 3 or end with 5.

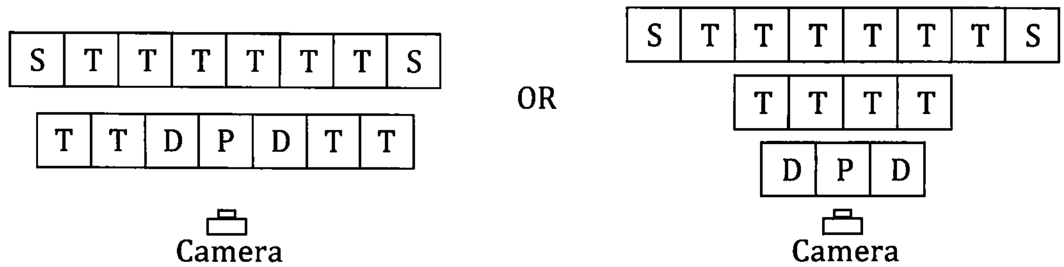
5. (a) How many eight letter arrangements are there of the letters of the word FORECAST if in each arrangement each letter must be used just once?
   How many of these arrangements
   (b) have the E and the O next to each other,
   (c) have the E and the O separated,
   (d) start with A and have E and O next to each other,
   (e) start with A or end with S?

6. Three digit numbers are to be made using the digits 1, 2, 3, 4, 5, 6, 7 with no digit used more than once in any particular three digit number.
   (a) How many three digit numbers are possible?
   How many of these
   (b) start with 4,
   (c) end with 5,
   (d) start with 4 and end with 5,
   (e) start with 4 or end with 5,
   (f) are odd numbers,
   (g) are greater than 700,
   (h) are greater than 500,
   (i) are even and greater than 500?

7. Four members of a sports team have to arrange themselves in a line for a photograph.
   The members are: Terri, Jen, Diane, May.
   How many arrangements are there in which
   (a) Terri is at the left end,
   (b) Diane is at the right end,
   (c) Terri is at the left end and Diane is at the right end,
   (d) Terri is at the left end or Diane is at the right end,
   (e) Jen and Diane occupy the middle two positions,
   (f) Jen and Diane are not next to each other?

8. (a) How many seven digit odd numbers can be made using the digits 0, 1, 2, 3, 4, 5, 6, 7 and 8 if no digit may be used more than once in a number and zero cannot be the first digit.
   (b) How many of these numbers from (a) are less that 4 000 000?

9. The staff at a particular college consist of 1 principal (P), 2 deputies (D), 10 teachers (T) and 2 support staff (S). The two styles of arranging the staff that are being considered for the staff photograph are shown below.



How many different arrangements of the 15 staff are there?

10. To make a two letter code the first letter is chosen from the set of vowels and the second letter is chosen from the 26 letters of the alphabet.
    (a) How many different codes are possible?
    How many of the codes
    (b) start with an E,
    (c) end with a D,
    (d) start with an E and end with a D,
    (e) start with an E or end with a D,
    (f) start and end with the same letter,
    (g) consist of two different letters?

Combinations.

There are $4 \times 3 \times 2$, i.e. ${}^4P_3 = 24$, arrangements of three letters taken from the letters A, B, C and D:

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

Suppose now that A, B, C and D represented four people, Alex, Bonnie, Chris and Dan and that we are wishing to select three of these four to attend a meeting for which the order of the three is unimportant. There are now just four **selections,** or **combinations** of three items chosen from four different items:

ABC	ABD	ACD	BCD

A combination is a selection – the order does not matter.
A permutation is an arrangement – the order does matter.

We use the notation ${}^nC_r$ for the number of **combinations** of r different objects taken from a set of n different objects.

In the example of choosing a group of 3 letters from 4 letters our initial listing of 24 arrangements reduced to just 4 selections because, when order became unimportant, each column of 6 (= 3!) arrangements reduced to just one selection.

$$\begin{aligned} {}^4C_3 &= \frac{{}^4P_3}{3!} \\ &= \frac{4!}{(4-3)!\ 3!} \end{aligned}$$

To apply this thinking to combinations of r objects chosen from n different objects

$$\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n!}{(n-r)!\ r!} \end{aligned}$$

There are ${}^nC_r$ combinations of r objects chosen from n different objects where

$${}^nC_r = \frac{n!}{(n-r)!\,r!}.$$

Example 15
How many combinations are there of two objects chosen from five different objects that we will call A, E, I, O and U.

$$\begin{aligned} \text{Number of combinations} &= {}^5C_2 \\ &= \frac{5!}{(5-2)!\,2!} \\ &= \frac{5!}{3!\,2!} \\ &= 10 \end{aligned}$$

```
nCr(5,2)
                10
```

The ten combinations are:

AE,	AI,	AO,	AU,
EI,	EO,	EU,	
IO,	IU,		
OU.			

Note
- ${}^{n}C_r$ is also written $\binom{n}{r}$. For example $\binom{7}{2} = {}^{7}C_2$
- ${}^{n}C_r$ can be thought of as "from n *choose* r".
- If asked the question "How many ways can five people be arranged in a line for a photograph given that the five can themselves be selected from a larger group of eight" we can solve this as before:

  $$\begin{aligned}\text{Number of arrangements} &= 8 \times 7 \times 6 \times 5 \times 4 \\ &= 6\,720\end{aligned}$$

  or we could consider this as "from 8 people choose 5 and then arrange them".

  $$\begin{aligned}\text{Number of arrangements} &= {}^{8}C_5 \times 5! \\ &= 6\,720\end{aligned}$$

Example 16
How many combinations are there of five people to attend a particular conference if the five are to be selected from twelve people?

From 12
choose 5

$$\begin{aligned}\text{Number of combinations} &= {}^{12}C_5 \\ &= \frac{12!}{(12-5)!\ 5!} \\ &= \frac{12!}{7!\ 5!} \\ &= 792\end{aligned}$$

Example 17
How many different ways can a group of six people, 3 male and 3 female, be selected from 8 males and 9 females?

	Male			Female
From	8	and	from	9
choose	3		choose	3

$$\begin{aligned}\text{Number of ways} &= {}^{8}C_3 \times {}^{9}C_3 \\ &= 4\,704\end{aligned}$$

A group of three males and three females can be selected from eight males and nine females in 4 704 ways.

Note that in ${}^{8}C_3 \times {}^{9}C_3$ the upper numbers add to 17 and the lower to 6. This makes sense because we are choosing 6 people from 17 but we are being careful to select the correct number of people from each subgroup.

Example 18

A normal pack of playing cards consists of 52 cards arranged in four suits.

Hearts (red)	A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
Diamonds (red)	A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
Spades (black)	A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
Clubs (black)	A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣

These cards are well shuffled and 7 cards are randomly dealt to form a "hand".

(a) How many different 7 card hands are there?

How many of these hands contain:

(b) the ace of hearts (A♥),
(c) the ace and two of hearts (A♥ and 2♥),
(d) exactly 3 of the 4 kings,
(e) at least one ace?

It is the cards that make up the hand that is important, not the order in which they are dealt. Thus this situation involves combinations.

(a)

From	52	
Choose	7	

Number of hands $= {}^{52}C_7$
$= 133\,784\,560$

(b)

	A♥	Others
From	1	51
Choose	1	6

Number of hands $= {}^{1}C_1 \times {}^{51}C_6$
$= 18\,009\,460$

(c)

	A♥	2♥	Others
From	1	1	50
Choose	1	1	5

Number of hands $= {}^{1}C_1 \times {}^{1}C_1 \times {}^{50}C_5$
$= 2\,118\,760$

(d)

	K	Others
From	4	48
Choose	3	4

Number of hands $= {}^{4}C_3 \times {}^{48}C_4$
$= 778\,320$

(e) Number of hands with no aces:

	A	Others
From	4	48
Choose	0	7

Number of hands $= {}^{4}C_0 \times {}^{48}C_7$
$= 73\,629\,072$

Number with at least one ace $= 133\,784\,560 - 73\,629\,072$
$= 60\,155\,488$

Note carefully the use of the complementary event in part (e).

Alternatively we could consider hands with 1 ace + hands with 2 aces + etc. i.e.
${}^{4}C_1 \times {}^{48}C_6 + {}^{4}C_2 \times {}^{48}C_5 + {}^{4}C_3 \times {}^{48}C_4 + {}^{4}C_4 \times {}^{48}C_3 = 60\,155\,488$, as before.

Example 19

A subcommittee of five people is to be chosen from the following twelve people:

Alex	Ben	Chris	Dave	Eric	Frank
Gemma	Hetti	Icolyn	Jenny	Kym	Louise

How many different subcommittees are possible in each of the following cases.

(a) There are no restrictions as to the make up of the subcommittee.

(b) Jenny and Eric must both be on the subcommittee?

(c) Ben and Gemma must either both be on the subcommittee or neither be on the subcommittee.

(d) Dave and Icolyn must not both be on the subcommittee. Dave can be, or Icolyn can be, but not both.

(a)

From	12
Choose	5

$$\text{N}^{\text{o.}}\text{ of subcommittees} = {}^{12}C_5 = 792$$

(b)

	J & E	Others
From	2	10
Choose	2	3

$$\text{N}^{\text{o.}}\text{ of subcommittees} = {}^{2}C_2 \times {}^{10}C_3 = 120$$

(c) <u>B and G both on</u> OR <u>B and G both not on</u>

	B & G	Others
From	2	10
Choose	2	3

	B & G	Others
From	2	10
Choose	0	5

$$\text{N}^{\text{o.}}\text{ of subcommittees} = {}^{2}C_2 \times {}^{10}C_3 + {}^{2}C_0 \times {}^{10}C_5 = 372$$

(d)

	D not I			I not D			Neither		
	D	I	Others	D	I	Others	D	I	Others
From	1	1	10	1	1	10	1	1	10
Choose	1	0	4	0	1	4	0	0	5

$$\text{N}^{\text{o.}}\text{ of subcommittees} = {}^{1}C_1 \times {}^{1}C_0 \times {}^{10}C_4 + {}^{1}C_0 \times {}^{1}C_1 \times {}^{10}C_4 + {}^{1}C_0 \times {}^{1}C_0 \times {}^{10}C_5 = 672$$

Alternatively, for part (d), we could find the number of committees containing both Dave and Icolyn and take this from the total number of subcommittees:

$$\text{N}^{\text{o.}}\text{ of subcommittees} = {}^{12}C_5 - {}^{2}C_2 \times {}^{10}C_3 = 792 - 120 = 672 \text{ as before.}$$

Example 20

How many different 5 letter arrangements can be made each consisting of 5 different letters of the alphabet, with exactly one of the 5 being a vowel?

Method 1: Suppose the vowel is first.

Number of arrangements with vowel first $= 5 \times 21 \times 20 \times 19 \times 18$
$= 718\,200$

But the vowel could be in any of the five positions

Number of arrangements with one vowel $= 5 \times 718\,200$
$= 3\,591\,000$

Method 2:

Choose one vowel.	Number of ways	$= {}^5C_1$
Choose four consonants.	Number of ways	$= {}^{21}C_4$
Arrange the five items.	Number of ways	$= 5!$

Number of arrangements with one vowel $= {}^5C_1 \times {}^{21}C_4 \times 5!$
$= 3\,591\,000$ as before.

- Note especially method 2 above in which the number of arrangements have been determined using a "choose and then arrange" approach.

Example 21

Including A itself and the empty set how many subsets can be made using the elements of set A where A = {a, e, i, o, u}.

There exists

1 subset of A with no elements. ${}^5C_1$ subsets with 1 element.

${}^5C_2$ subsets with 2 elements. ${}^5C_3$ subsets with 3 elements.

${}^5C_4$ subsets with 4 elements. ${}^5C_5$ subsets with 5 elements.

Total number of subsets $= 1 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$
$= 1 + 5 + 10 + 10 + 5 + 1$
$= 32$

Alternatively, when forming a subset of {a, e, i, o, u} we can consider each element on an "include it or not include it" basis. We can include the letter a or not, we can then include the letter e or not, we can then include the letter i or not, etc. Thus there are 2 ways of dealing with each letter.

Thus number of subsets $= 2 \times 2 \times 2 \times 2 \times 2$
$= 2^5$
$= 32$

> If we include the empty set and the set itself then a set with n different elements has 2^n possible subsets.

Example 22

How many three letter permutations are there of the letters of the word PARALLEL ?

Parallel involves 8 letters including 2 As and 3 Ls.

P	A	R	L	E
	A		L	
			L	

Consider mutually exclusive situations:

Arrangements with

3 letters, all different:	$5 \times 4 \times 3$	=	60
2 As and one other, AA?. A?A, ?AA	3×4	=	12
2 Ls and one other, LL?. L?L, ?LL	3×4	=	12
3 Ls	1	=	1
	60 + 12 + 12 = 1	=	85

There are 85 permutations altogether.

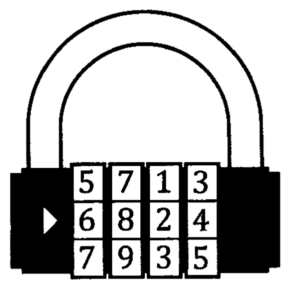
Or, using a "choose and then arrange" approach:

Arrangements with

3 letters, all different:	${}^5C_3 \times 3!$	=	60
2 As and one other, AA?. A?A, ?AA	${}^4C_1 \times \frac{3!}{2!}$	=	12
2 Ls and one other, LL?. L?L, ?LL	${}^4C_1 \times \frac{3!}{2!}$	=	12
3 Ls	1	=	1
	60 + 12 + 12 = 1	=	85

There are 85 permutations altogether, as before.

Exercise 2E



1. Is a combination lock correctly named?

2. How many combinations are there of 3 different letters chosen from the set {a, b, c, d, e}?

3. How many combinations are there of four people to attend a particular conference if the four are to be selected from twenty people?

4. How many different ways can a group of six people, 3 male and 3 female, be selected from 7 males and 10 females?

5. How many different soccer teams each consisting of 1 goalkeeper, 3 defenders, 4 midfielders and 3 strikers are possible if selection is made from 2 goalkeepers, 5 defenders, 7 midfielders and 3 strikers?

6. A college offers its year 11 mathematics students a choice of twelve units:

List I	List II	List III	List IV
Calculus I. Statistics I.	Vectors. Trigonometry. Matrices.	Equations. Networks. Optimisation. Sets.	Correlation. Time series. Counting.

Students must choose six units in all, the two from list I, one from list II, one from list III and two from list IV.
How many different allowable six unit combinations are there?

7. Four people are selected from a committee of 12 to represent the committee at a particular function. How many different groups of four are possible?
How many groups of four are possible if either the chairperson or the vicechairperson, but not both, must be in the group?

8. Including A itself and the empty set how many subsets can be made using the elements of set A where A = {a, b, c, d, e, f, g}.

9. If we define the *proper* subsets of a set as those subsets other than the empty set and the set itself, how many proper subsets are there for the set
{1, 2, 3, 4, 5, 6, 7, 8, 9}?

10. A company wishes to send a team of seven people overseas to investigate possible markets for its products. The team is to comprise 1 manager, 1 engineer, 3 marketing people and 2 legal experts. These are chosen from 3 managers, 12 engineers, a marketing team of 15 and a legal team of 5.
How many different such teams of seven are possible?
Joe is one of the 12 engineers and Sue is one of the 5 in the legal team.
How many of the possible teams include (a) Joe,
(b) Sue,
(c) at least one of these 2 people?

11. A subcommittee of four people is to be chosen from the following ten people:

Alice	Betty	Connie	Daisy	Ennis
Fred	Gerry	Henry	Indi	Javed

How many different subcommittees are there in <u>each</u> of the following cases?
(a) There are no restrictions as to the make up of the subcommittee.
(b) Javed and Ennis must both be on the subcommittee?
(c) Connie and Fred must either both be on the committee or neither of them be on the subcommittee.
(d) Betty and Henry must not both be on the subcommittee. Betty can be, or Henry can be, but not both.

12. A group of seven people is to be formed from a larger group comprising 8 women and 6 men. How many different groups of seven people are there if the group must consist of
    (a) 4 women and 3 men,
    (b) all women,
    (c) all men,
    (d) more than five women,
    (e) more men than women?

13. A normal pack of playing cards consist of 52 cards arranged in four suits.

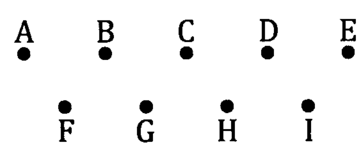
Hearts (red)	A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
Diamonds (red)	A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
Spades (black)	A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
Clubs (black)	A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣

Suppose these cards are well shuffled and eight cards are randomly dealt to form a "hand", the order the cards are dealt being irrelevant.
(a) How many different 8 card hands are there?

How many of these hands contain:
(b) the jack of hearts (J♥), (c) 5 red cards and the rest black,
(d) exactly two queens, (e) at least two queens.

14. Four places exist on a course for fire officers. The four are to be chosen from 13 officers: 6 from A division, 4 from B division and 3 from C division.
How many different groups of four are there?
How many different groups of four are there if each group must contain at least one officer from each division?

15. Points A, B, C, D, E, F, G, H and I are arranged in two rows as shown on the right. How many triangles can be formed having vertices chosen from these points?
How many of these triangles have point A as one of their vertices?



16. How many three letter permutations are there of the letters of the word
EQUILATERAL ?

17. How many four letter permutations are there of the letters of the word
CANDLEPOWER ?

$^{n}C_{r}$ and Pascal's triangle.

The arrangement of numbers shown below is known as Pascal's triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Notice that each line starts and finishes with a 1 and other entries are obtained by adding the two numbers "above left" and "above right".

$$\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & & & \swarrow & & \searrow & & & & & & & & & \\
 & & & & & & & & 1 & & & & 1 & & & & & & & & \\
 & & & & & & & \swarrow & & \searrow & & \swarrow & & \searrow & & & & & & & \\
 & & & & & & 1 & & & & 2 & & & & 1 & & & & & & \\
 & & & & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & & & & \\
 & & & & 1 & & & & 3 & & & & 3 & & & & 1 & & & & \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & & \\
 & & 1 & & & & 4 & & & & 6 & & & & 4 & & & & 1 & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\
1 & & & & 5 & & & & 10 & & & & 10 & & & & 5 & & & & 1
\end{array}$$

Not immediately obvious is the fact that the entries in Pascal's triangle can be expressed in $^{n}C_{r}$ notation:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & ^{1}C_{0} & & ^{1}C_{1} & & & & \\
 & & & ^{2}C_{0} & & ^{2}C_{1} & & ^{2}C_{2} & & & \\
 & & ^{3}C_{0} & & ^{3}C_{1} & & ^{3}C_{2} & & ^{3}C_{3} & & \\
 & ^{4}C_{0} & & ^{4}C_{1} & & ^{4}C_{2} & & ^{4}C_{3} & & ^{4}C_{4} & \\
^{5}C_{0} & & ^{5}C_{1} & & ^{5}C_{2} & & ^{5}C_{3} & & ^{5}C_{4} & & ^{5}C_{5}
\end{array}$$

Exercise 2F

1. Use the fact that $^{n}C_{r} = \dfrac{n!}{(n-r)!\ r!}$ to show that (a) $^{n}C_{r} = {}^{n}C_{n-r}$,

   (b) $\dfrac{n}{r} \times {}^{n-1}C_{r-1} = {}^{n}C_{r}$.

2. Show that the way in which the entries in Pascal's triangle are formed by adding "above left" and "above right" is consistent with the following statement:

   $$^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$$

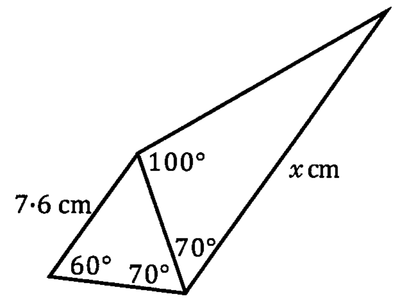
   Use the fact that $^{n}C_{r} = \dfrac{n!}{(n-r)!\ r!}$ to show that the above statement is true.

Miscellaneous Exercise Two.

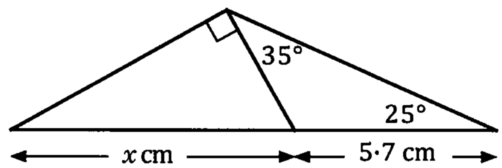
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

For each of questions 1 and 2 use trigonometry to determine x, correct to one decimal place, clearly showing the use of trigonometry in your working. (Diagrams not to scale.)

1.



2.



3. Given the true statement If $x = 8$ then $x^2 = 64$
write both the converse statement and the contrapositive statement and for each one state whether true or false.

4. As an introduction to counting techniques Ms Jackson, a Mathematics teacher, gives each of the twenty five students in her class a piece of paper and asks each to write down a permutation of the letters of the word FISH that uses all four letters once and once only.
Every one of the students performs this class correctly and Ms Jackson then collects in the 25 pieces of paper.
What can we conclude about the responses, and why?

5. If you did question 14 in exercise 2B you encountered Sanshi, at the race track, wanting to place just one more bet, either on race 8 or race 9. In that question he was not sure whether to randomly predict 1st, 2nd and 3rd for race 8, in which there were 8 horses racing or randomly predict a 1st and 2nd for race 9 with 12 horses racing.
Suppose instead that he makes his one bet that of attempting to randomly select the correct order for all five horses, i.e. 1st in race 8, 2nd in race 8, 3rd in race 8, 1st in race 9, 2nd in race 9. How many different bets of this type are there?

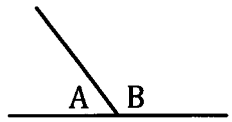
6. How many arrangements of five different letters can be made from the letters a, b, c, d, e, f, g, h and i?
How many of these arrangements contain 2 vowels and 3 consonants?
(Hint: *Choose* 2 vowels then *choose* 3 consonants and then *arrange* the 5 letters.)

7. A child is told she can bring five toys with her on holiday. The child decides to choose the five from
6 jigsaws, 8 dolls, 4 balls, 2 trucks.
How many different sets of five toys are there?
How many of these sets have at least one from each of the four categories of toy listed above?

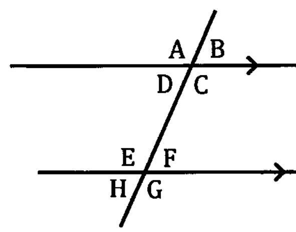
Questions 8 to 11 each ask you to prove some geometrical fact. You should set out your proofs as in the examples given in the Preliminary Work section involving similar triangles and congruent triangle. I..e draw a clear diagram, state what you are given, what you have to prove and any constructions made. Then set out your proof clearly and with statements justified.

In your proofs the following may be stated as fact, without proof:

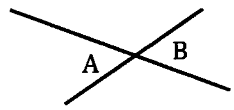
☛ Angles that together form a straight line have a sum of 180°.
I.e., in the diagram on the right, A + B = 180°.
(And conversely, if the angles have a sum of 180° then they together form a straight line.)



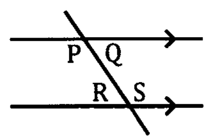
☛ When a transversal cuts parallel lines, corresponding angles are equal.
I.e., in the diagram on the right, A = E,
B = F,
C = G,
D = H.



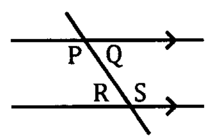
☛ When two straight lines intersect the vertically opposite angles are equal. I.e., in the diagram on the right, A = B.



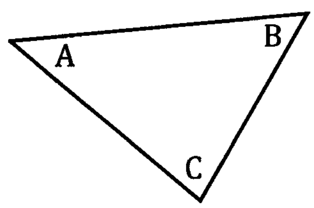
☛ When a transversal cuts parallel lines, alternate angles are equal. I.e., in the diagram on the right, Q = R,
P = S
(And, conversely, if alternate angles are equal we have parallel lines.)



☛ When a transversal cuts parallel lines, co-interior angles are supplementary. I.e., in the diagram on the right, P + R = 180°,
Q + S = 180°.



☛ The angles of a triangle add up to 180°.
I.e., in the diagram on the right, A + B + C = 180°.



8. Prove that if a quadrilateral has its opposite sides parallel then its opposite sides must be of equal length.

9. Given that a parallelogram has opposite sides parallel and equal prove that the diagonals of the parallelogram bisect each other.

10. If a quadrilateral has all four sides of equal length we call it a rhombus.
Prove that a rhombus must also have its opposite sides parallel (i.e that a rhombus must also be a parallelogram).

11. Prove that if two lines, AB and CD, bisect each other at right angles then:
AD = DB = BC = CA.